Convection

It is a **thermal** instability that allows for energy transport through macroscopic motion of matter, resulting in a non-zero energy flux and a net-zero matter flux.

Q: what are the energy transport mechanism in stars?
A:
- radiation (photons carry energy)
- convection
- conduction (electrons carry energy, important eg. in WDs)

We can define a timescale for convection:

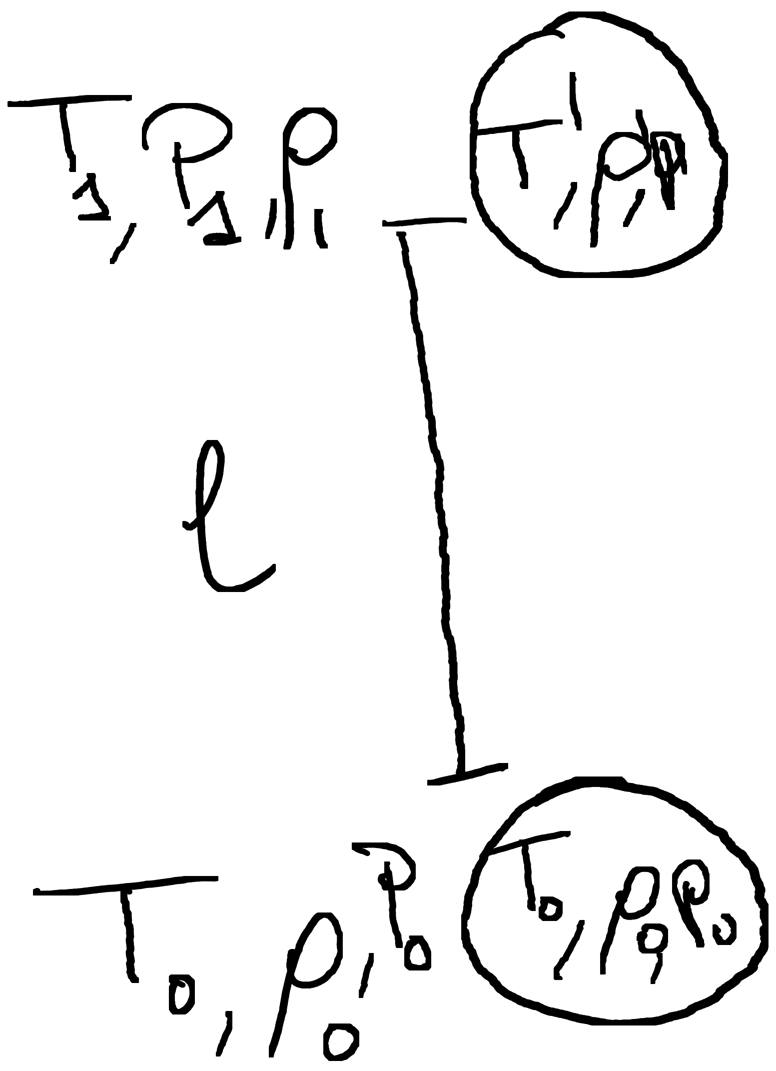
convective turnover timescale: $\tau_{conv} \simeq \Delta R / v_{conv}$

Q: To what timescale of the star do you think this will be proportional to? (free-fall? thermal? nuclear?)
A: Since convection is a thermal instability, $\tau_{conv} \propto \tau_{thermal}$ with a small proportionality constant.

Convective stability: the Schwarzschild criterion

Given that convection is an instability, we expect it to develop when other energy transport means are insufficient to carry the energy flux. While we still have many open questions on how convection turns on/off and on how to properly model it, we can easily work out a STABILITY CRITERION.

N.B. I will focus on the simplest case of homogeneous composition although there are current debates on which stability criterion to consider in case of compositional gradients.



The bubble picture:

to derive a stability criterion, let's perturb an element of gas and see how it reacts.

Initially it's in equilibrium with the environment, if we move it (vertically) by ℓ, the background at the new location has T_1, ρ_1 and the bubble has T', ρ' yet unknown.

Since convection is a thermal instability, we can assume hydrostatic balance between the bubble and the environment. If this weren't the case, the pressure difference would lead to expansion/contraction at the sound speed to restore pressure balance on a sound-crossing (i.e. dynamical) timescale.

$$\Rightarrow P' = P_1 \qquad (\text{or } P_{bubble} \equiv P_{environment} \text{ throughout})$$

Now let's calculate the density. We can assume ℓ is small, and to 1st order:

$$\rho' = \rho_0 + \left.\frac{d\rho}{dz}\right|_{ad} \ell$$

Where we use the ADIABATIC gradient assuming the bubble does not exchange energy with the environment. For the environment

$$\rho_1 = \rho_0 + \left.\frac{d\rho}{dz}\right|_{env} \ell \equiv \rho_0 + \left.\frac{d\rho}{dz}\right|_{rad} \ell$$

Because we assume radiative transport dominates in the background before (eventually, if needed) convection develops

INSTABILITY CONDITION: the buoyancy would continue to push the bubble further up

This requires the bubble to be less dense than the environment after the displacement, or in other words latter

$$\rho' < \rho_1 \Rightarrow \boxed{\left.\frac{d\rho}{dz}\right|_{ad} < \left.\frac{d\rho}{dz}\right|_{env} \simeq \left.\frac{d\rho}{dz}\right|_{rad}}$$

Using an equation of state $P \equiv P(\rho, T, \mu)$ we can rewrite this in terms of temperature gradients, and it is typically convenient to use P instead of z as independent variable. After some algebra, assuming homogeneous chemical composition:

$$\boxed{\text{CONVECTIVE INSTABILITY} \iff \nabla_{ad} < \nabla_{rad}}$$

Schwarzschild criterion

with $\nabla_i = \left.\dfrac{\partial \log(T)}{\partial \log(P)}\right|_i$

If one allows for variations in composition, there is another term on the left ($\nabla_\mu = \partial \log(\mu) / \partial \log(P)$)

$$\boxed{\text{CONVECTIVE INSTABILITY} \iff \nabla_{ad} + \frac{\varphi}{\delta} \nabla_\mu < \nabla_{rad}}$$

Ledoux criterion

Although there is now debate on whether on timescales of thousands of convective turnover timescales these differ.

If a layer is Ledoux-stable but Schwarzschild unstable we call the resulting instability SEMICONVECTION

While Ledoux-unstable but Schwarzschild stable is referred to as thermohaline mixing.

Note that these can occur (maybe) in stars, but also atmosphere and Oceans on Earth.

Take home point

⇒ If the density gradient in the star becomes too steep (i.e. steeper than adiabatic), then radiative diffusion cannot carry the energy flux and any small perturbation will result in a convective instability

Convective energy flux: to calculate that, we need to solve for the dynamics of the bubbles, which happens on timescales related to the thermal timescale: it's not even really possible in general.
Erika Böhm-Vitense (1957) developed to this and a MEAN FIELD THEORY TO DESCRIBE THE SPACE- and TIME-AVERAGED state reached by a convective layer AFTER THE INSTABILITY SATURATES

Example: convection is what allows self-sustained flames (convective motion refuels of oxygen the layer where oxydoreduction reactions happen)

MLT is like averaging in time and space the flame

Assuming some ergodicity, it's like describing the spatial average of a picture of a cross section of the flame.

Q: Before going into the details you can maybe guess where this model is insufficient?
- timescales short compared to τ_{conv} (no time to reach the average steady state described by MLT)
- convective boundaries (far from average layers)

To calculate F_{conv}, let's consider the excess temperature of a "bubble" (i.e. a thermal flux tube), which raises adiabatically w.r.t. the environment:

$$T_{bubble} - T_{env} \simeq \left(T_0 + \frac{dT}{dr}\bigg|_{bubble} \ell\right) - \left(T_0 + \frac{dT}{dr}\bigg|_{env} \ell\right) = (\nabla_{ad} - \nabla_{env}) \cdot \ell$$

Multiplying by the specific heat and density, we obtain the thermal energy difference carried by the "bubble" (per unit volume)

$$\Delta E_{thermal}/_{Vol} = c_p \rho \, (\nabla_{ad} - \nabla_{env}) \ell$$

Q: why do we use the specific heat at constant pressure c_p?
→ We assume pressure equilibrium between the bubble and environment (even if P changes across the bubble travel path ℓ)

And multiplying by the bubble velocity (the "convective velocity", which is the average speed of thermal energy flux tubes) we obtain the heat flux carried by convection:

$$F_{conv} = c_p \rho (\nabla_{ad} - \nabla_{env}) \ell v_{conv}$$

However, we are taking $(\nabla_{ad} - \nabla_{env})$ across the whole travel path ℓ, on average the temperature difference is only $\sim 1/2$ of that. But, on average for each bubble raising and finding itself hotter, there will be one sinking and finding itself colder, thus extracting heat from the surrounding and contributing the same amount to the flux.

We have 2 poorly known things: the distance travelled by the bubbles on average (before they transfer their excess/defect in thermal energy and dissolve), and their velocity.

To calculate the convective velocity, we can calculate the work done by the buoyancy on the bubble.

The density difference of the bubble to environment is

$$\rho_{bubble} - \rho_{env} = \left(\rho_0 + \frac{d\rho}{dr}\bigg|_{ad} \ell\right) - \left(\rho_0 + \frac{d\rho}{dr}\bigg|_{env} \ell\right)$$ EOS and definition of ∇_i

$$= \frac{\rho}{T} (\nabla_{ad} - \nabla_{env}) \ell$$

$\Rightarrow$ Multiplying by $\frac{1}{2} g$ we get the average buoyancy force acting on the bubble across the travel

$$B \simeq \frac{1}{2} g \frac{\rho}{T} (\nabla_{ad} - \nabla_{env}) \ell$$

N.B: $g \equiv g(r) = \frac{-Gm(r)}{r^2}$

and $\vec{B} \cdot \vec{\ell}$ is the work done on the bubble, which by energy conservation is equal to the bubble's kinetic energy (because of the assumption $\nabla_{bubble} = \nabla_{ad}$ there are no heat exchanges during the adiabatic motion of the bubble):

$$E_{kin, bubble} = \vec{B} \cdot \vec{\ell} \Rightarrow \frac{1}{2} \rho v_{conv}^2 = \frac{\rho}{T} (\nabla_{ad} - \nabla_{env}) \ell^2 \frac{Gm(r)}{r^2}$$

N.B: $\vec{B}$ and $\vec{\ell}$ are antiparallel

Q: upper limit for v_{conv}? Sound speed! otherwise pressure balance assumption ...

Note that V_{conv} and ℓ appear squared, so this hold both upward and downward for bubbles. That relates V_{conv} and ℓ, leaving only one unknown for the convective flux: THE AVERAGE DISTANCE TRAVELLED BY A BUBBLE before it loses is excess heat, aka MIXING LENGTH THEORY.

Erika Bohm-Vitense (1957) proposed

$$\ell = \alpha H_p \quad \text{with } H_p = -\frac{dr}{d\log(P)}$$

and α the famous free parameter for the Mixing length theory, which means the bubble travel a portion of the pressure scale height.

Q: The boiling pasta pot: is it really convection? What carries the heat? Radiation is not the most important, conduction also matters at the beginning. However, the functional form of the conductive and radiative temperature gradient is the same (up to a change in opacity κ): indeed at some point the pot becomes convectively unstable. But this is NOT the bubbles!! The bubbles in that pot are de to the water liquid/vapor phase transition. You can typically see convective flux as little wiggles just before the phase transition kicks in.

Summary of key assumptions of Mixing Length Theory:

- environmental gradient is radiative ($\nabla_{env} = \nabla_{rad}$)
- we model thermal flux tubes that carry a net thermal energy flux but a net zero mass flux as bubbles rising ADIABATICALLY and at instantaneous pressure equilibrium with the environment that travel $\ell = \alpha H_p$ (on average) and with $\nabla_{bubble} = \nabla_{ad}$
- this allows for a description of the average mean energy flux carried by convection, where the convective flux is proportional to the superadiabaticity $\nabla_{ad} - \nabla_{rad}$.

$$F_{conv} = \frac{1}{4} c_p \rho \left(\frac{G m(r)}{T r^2}\right)^{1/2} (\nabla_{ad} - \nabla_{rad})^{3/2} \ell^2$$

CONVECTIVE FLUX FROM MLT

Superadiabaticity and the efficiency of convection

Convection is a rather strong instability, that is extremely efficient at carrying energy. As a result, the typical superadiabaticity in stars is very small:

$$\nabla_{ad} - \nabla_{rad} \lesssim 10^{-8} \qquad \text{N.B. } \left.\frac{d\log(T)}{d\log(P)}\right|_i \text{ is dimensionless}$$

$\Rightarrow$ whenever convection is EFFICIENT the actual temperature gradient in a convectively unstable layer is approximately adiabatic.

This is typical in stellar interior (eg. massive stars core). At the surface, where ρ is low, convection may be inefficient and the superadiabaticity is not always negligible.

The elephant in the room: TURBULENCE

Allegedly, Erika Bohm Vitense said if she were aware of Kolmogorov work on turbulence, she would not have proposed MLT (it was during the cold war!)

A layer is turbulent for large Reynolds numbers

$$R_e = \frac{\ell \, v_{conv}}{\nu}$$

Since in stars ν (microscopic viscosity) is small, and in stellar interiors $\ell = \alpha H_p$ is very large, even for $v_{conv} = c_s$ (for which MLT would not hold, in particular the assumption of HSE) is typically enormous:

TURBULENT FLOWS ARE SUBSONIC BUT VERY TURBULENT, the "bubble picture" IS AN OVERSIMPLIFICATION to describe the mean-field, time-average, thermal energy flux carried by the complex turbulent flow.

Recall: radiative temperature gradient

$$\frac{dT}{dr} \propto \kappa \frac{L}{4\pi r^2} = \kappa F_{rad}$$

$\Rightarrow$ convection occurs where dT/dr is steep (er than adiabatic), which can occur if the radiative opacity κ is high (photons are too impeded to efficiently carry energy) or if F_{rad} is too large (eg. next to where energy is released).

Q: why is the Sun's envelope convective? Or a protostar? why in a RSG? Why the core of a massive star is? What if κ locally increases in the star (eg. because of recombination)

$\longleftrightarrow$

References:

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Schwarzschild, chapter 7

Jermyn et al. 2023

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Kippenhan, Weigert, Weiss, chapter 6

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